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On Space-Time Permeated by the Perfect Magnetofluid

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Abstract

A space-time permeated by the self-gravitating perfect fluid with infinite electrical conductivity and constant magnetic permeability (perfect magnetofluid) is investigated. For a C space defined as the space in which the divergence of conformal curvature vanishes, it is proved that the rotation explicitly depends on the magnetic field. In a J space characterized by the vanishing of the divergence of Petrov space-matter tensor, the invariance of the energy density, the isotropic pressure, and the magnitude of the magnetic field along the divergence-free magnetic lines is established. It is found that if the stress-energy tensor of the perfect magnetofluid is a Killing tensor, the energy density, the isotropic pressure, and the magnitude of the magnetic field are constant. Moreover it is shown that the stream lines are expansion-free and the magnetic lines are divergence-free. It is proved that the complexion of the field of the perfect magnetofluid remains invariant along the magnetic lines if and only if these lines are normal to the lines of vorticity.

1. *Introduction*

Field equations for thermodynamical perfect fluid with infinite electrical conductivity and constant magnetic permeability (we call such a fluid a perfect magnetofluid) were announced by Lichnerowicz (1967). He had established the existence and uniqueness of their solutions. His field equations were used by Ozsvath (1967) to obtain two families of Lichnerowicz universes with geodesic flow and shear-free fluid. The absolute derivative of the nonmagnetic proper energy density and that of the magnetic field were used by Yodzis (1971) to study the effect of the magnetic field on gravitational collapse and the orientation of the magnetic field in intergalactic space and pulsars. On assuming the variable magnetic permeability, Date (1972) had shown explicit dependence of entropy generation on the magnetic field. Date (1973a, b) and Bray (1971, 1972a, b) have obtained exact solutions of the Lichnerowicz field equations. Definite material schemes and P spaces for perfect magnetofluid were investigated by Shaha (1974). Recently, Date (1976) studied the gravitational field in perfect magnetofluid by finding Maxwell-like equations for the gravitational field in the magnetofluid and the expressions for the refractive index and the ray shear of a null gravitational field. A set of necessary and sufficient conditions is obtained by Asgekar and Date (1976) in the case of the prefect magnetofluid admitting "Einstein collineation" with respect to the preferred directions.

Geometry of the space-time permeated by the magnetofluid is a four-dimensional Riemannian manifold V_4 with the metric g_{ab} and four-velocity u^a. The signature of the metric is $(-, -, -, +)$. Latin indices run from 1 to 4. The semicolon indicates covariant differentiation and comma indicates partial differentiation. Symmetrization and antisymmetrization are denoted by round brackets and square brackets around the suffixes, respectively. Units are such that the gravitational constant and the velocity of light are one.

The covariant derivative of u^a can be decomposed as

$$u_{a;b} = \tfrac{1}{3}\theta p_{ab} + \dot{u}_a u_b + \sigma_{ab} + \omega_{ab} \tag{1.1}$$

where $\theta = u^a_{;a}$ is the expansion parameter, $\dot{u}^a = u^a_{;b}u^b$ is the four-acceleration, $\sigma_{ab} = u_{(a;b)} - \dot{u}_{(a}u_{b)} + \frac{1}{3}\theta p_{ab}$ is the shear tensor, $\omega_{ab} = u_{[a;b]} - \dot{u}_{[a}u_{b]}$ is the rotation tensor, and $p_{ab} = g_{ab} - u_a u_b$ is the three-space projection operator. The two-space projection operator is $q_{ab} = g_{ab} - u_a u_b + H_a H_b$, where H^a is a spacelike unit vector.

The concept of C space had been introduced by Szekeres (1964) as the space-time in which the divergence of the Weyl tensor C_{abcd} vanishes. The expression for the Weyl tensor C_{abcd} in terms of the Riemann curvature tensor R_{abcd} and the Ricci tensor $R_{ab} = R^c_{acb}$ is (Ellis, 1971)

$$C_{abcd} = R_{abcd} - g_{a[d}R_{c]b} - g_{b[c}R_{d]a} + \tfrac{1}{3}Rg_{a[d}g_{c]b} \tag{1.2}$$

The well-known Bianchi identities $R_{ab[cd;e]}$ imply (Szekeres, 1964)

$$C^d_{abc;d} = J_{abc} = R_{c[a;b]} - \tfrac{1}{6}g_{c[a}R_{,b]} \tag{1.3}$$

Thus a space-time is a C space if

$$J_{abc} = R_{c[a;b]} - \tfrac{1}{6}g_{c[a}R_{,b]} = 0 \tag{1.4}$$

A J space is characterized by the equation

$$R_{c[a;b]} = 0 \tag{1.5}$$

Consequently $R_{,b} = 0$, and $J_{abc} = 0$. Thus, every J space is a C space.

A second-order symmetric tensor S_{ab} is a Killing tensor if it satisfies the relation (Hauser and Malhiot, 1973)

$$S_{(ab;c)} = 0 \tag{1.6}$$

The three-space projection operator p_{ab} is a Killing tensor if

$$3p_{(ab;c)} = u_a u_{(b;c)} - u_b u_{(a;c)} - u_c u_{(a;b)} = 0 \tag{1.7}$$

The equations $p_{(ab;c)}u^a u^b = 0$ and $p_{(ab;c)}g^{ab} = 0$ yield

$$\dot{u}_a = 0, \qquad \theta = 0, \qquad \sigma_{ab} = 0 \tag{1.8}$$

The gradient of the complexion α has been expressed as (Singh et al., 1965)

$$\alpha_{,a} = \frac{\eta_a^{bcd} R^{\ e}_{b;c} R_{de}}{R_{lm} R^{lm}} \tag{1.9}$$

For the Einstein space that is characterized by $R_{ab} = \lambda g_{ab}$, equation (1.9) gives

$$\alpha_{,a} = 0$$

Thus, for the Einstein space, the gradient of the complexion is zero. For J space, defined by the equation (1.5), we observe that $\alpha_{,a} = 0$. Thus for the J space also the gradient of the complexion is zero.

2. *Field Equations and Differential Identities*

In the general theory of relativity the field equations for the thermodynamical perfect fluid with constant magnetic permeability and infinite electric conductivity are given by (Lichnerowicz, 1967) the Einstein equations

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = -T_{ab} \tag{2.1}$$

and the Maxwell equations

$$(u^a h^b - u^b h^a)_{;b} = 0 \tag{2.2}$$

where

$$T_{ab} = (\rho + p + \mu \mathbf{h}^2)u_a u_b - (p + \tfrac{1}{2}\mu \mathbf{h}^2)g_{ab} - \mu h_a h_b \tag{2.3}$$

Here ρ is the matter energy density, p is the isotropic pressure, μ is the magnetic permeability, u^a is the four-velocity vector, and h^a is the magnetic field vector satisfying the relations

$$u_a h^a = 0, \qquad h_a h^a = -\mathbf{h}^2$$

The equations connecting thermodynamical variables are

$$\rho = \rho_0(1 + \epsilon)$$
$$TdS = d\epsilon + pd(1/\rho_0) \tag{2.4}$$

where ρ_0 is the proper matter density, ϵ is the internal energy density, T is the rest temperature, and S is the specific entropy.

The local energy balance equation $T^{ab}_{;b} = 0$ produces the equations of the space-time trajectories of the magnetofluid particles in the form

$$(\rho + p + \mu \mathbf{h}^2)_{;b} u_a u^b + (\rho + p + \mu \mathbf{h}^2)(\dot{u}_a + \theta u_a) - (p + \tfrac{1}{2}\mu \mathbf{h}^2)_{,b}\delta_a^{\ b} - \mu(h_a h^b)_{;b} = 0 \tag{2.5}$$

Contractions of the Maxwell equations (2.2) yield

$$h^a_{;a} - h_{a;b}u^a u^b = 0 \tag{2.6}$$

$$h^a_{;a} + \dot{u}^a h_a = 0 \tag{2.7}$$

$$(\omega_{ab} + \sigma_{ab})h^b + h^b_{;b}u_a - \tfrac{2}{3}\theta h_a - h_{a;b}u^b = 0 \tag{2.8}$$

$$\sigma_{ab}h^a h^b + \tfrac{2}{3}\theta \mathbf{h}^2 + \tfrac{1}{2}\mathbf{h}^2_{,a}u^a = 0 \tag{2.9}$$

Using the Maxwell equations (2.2) in the time component of equation (2.5) we obtain the equation of continuity

$$\dot{\rho} + (\rho + p)\theta = 0 \tag{2.10}$$

The space component of (2.5) is given by

$$(\rho + p + \mu\mathbf{h}^2)\dot{u}^a - (p + \tfrac{1}{2}\mu\mathbf{h}^2)_{,b}p^{ab} - \mu(h^c h^b)_{;b}p_c{}^a = 0 \tag{2.11}$$

On using equations (2.4), equation (2.10) reduces to

$$x(\rho_0 u^a)_{;a} + \rho_0 TS_{,a}u^a = 0 \tag{2.12}$$

where $x = 1 + \epsilon + p/\rho_0$.

Consequences of the equations (2.6)-(2.9) the equations of motion (2.10) and (2.11), and the heat transfer equation (2.12) have been studied by Date (1976).

The equation $T^{ab}_{;b}\,h_a = 0$ with the Maxwell equations (2.2) yields

$$(\rho + p)\dot{u}^a h_a = p_{,a}h^a \tag{2.13}$$

Therefore, we have

$$\dot{u}^a h_a = 0 \Leftrightarrow p_{,a}h^a = 0 \qquad \text{as } (\rho + p) \neq 0 \tag{2.14}$$

Thus for the perfect magnetofluid, the isotropic pressure conserves along the magnetic field lines if and only if the four-acceleration is orthogonal to the magnetic field vector.

3. *C-Space and J-Space for the Perfect Magnetofluid*

It has been proved by Szekeres (1964) that the perfect fluid is a C space if and only if the stream lines are irrotational and shear-free with vanishing spatial density gradient. The defining expression (1.4) for the perfect magnetofluid supplies

$$Au_{a;[b}u_{c]} + Au_a u_{[c;b]} + A_{,[b}u_{c]}u_a + (\tfrac{1}{3}\rho + \tfrac{1}{2}\mu\mathbf{h}^2)_{,[c}g_{b]a} - \mu h_{a;[b}h_{c]} - \mu h_a h_{[c;b]} = 0 \tag{3.1}$$

where $A = \rho + p + \mu\mathbf{h}^2$.

Theorem 3.1. In C space for the perfect magnetofluid, rotation is only due to the magnetic field.

Proof. Contraction of equation (3.1) with u^a yields

$$Au_{[c;b]} + A_{,[b}u_{c]} + (\tfrac{1}{3}\rho + \tfrac{1}{2}\mu\mathrm{h}^2)_{,[c}u_{b]} - \mu u^a h_{a;[b}h_{c]} = 0 \qquad (3.2)$$

Further contracting equation (3.2) with $p_d{}^b p_e{}^c$, we get

$$A\omega_{de} = h^c(\dot{u}_c h_{[d}u_{e]} - u_{c;[d}h_{e]}). \qquad (3.3)$$

Thus

$$h^c = 0 \Leftrightarrow \omega_{de} = 0 \qquad (3.4)$$

and the proof of the theorem is complete.

Remark 1. C space for the perfect magnetofluid implies

$$(\rho + \tfrac{3}{4}\mu\mathrm{h}^2)_{,a}h^a = 0 \qquad (3.5)$$

Remark 2. The perfect magnetofluid with uniform magnetic field is irrotational [cf (3.3)].

Theorem 3.2. In J space for the perfect magnetofluid with a condition $\rho + p \neq \mu\mathrm{h}^2$, the magnitude of the magnetic field is conserved along the expansion-free flow.

Proof. For the perfect magnetofluid, equation (3.5) gives

$$A_{,[c}u_{b]}u_a + Au_{a;[c}u_{b]} + Au_a u_{[b;c]} + B_{,[b}g_{c]a} + \mu h_{a;[b}h_{c]} + \mu h_a h_{[c;b]} = 0 \qquad (3.6)$$

where $B = p + \tfrac{1}{2}\mu\mathrm{h}^2$. From equation (3.6) we get

$$(\rho - 3p)_{;a}u^a = 0 \qquad (3.7)$$

$$\mathrm{h}^2(p - \tfrac{1}{2}\mu\mathrm{h}^2)_{;a}u^a - (\rho + p)u_{a;b}h^a h^b = 0 \qquad (3.8)$$

Using equation of continuity (2.10) and equations (3.7) and (3.8) we obtain for $\theta = 0$

$$\tfrac{1}{2}\mu\mathrm{h}^2\mathrm{h}^2_{,a}u^a + (\rho + p)u_{a;b}h^a h^b = 0$$

i.e.,

$$(\rho + p - \mu\mathrm{h}^2)\mathrm{h}_{,a}u^a = 0 \qquad (3.9)$$

Thus $\mathrm{h}_{,a}u^a = 0$ when $\rho + p \neq \mu\mathrm{h}^2$, and the proof of the theorem is complete.

Theorem 3.3. In J-space for the perfect magnetofluid, the energy density, the isotropic pressure, and the magnitude of the magnetic field vector is invariant along the divergence-free magnetic lines.

Proof. From equation (3.6), we find

$$(\rho + \tfrac{1}{2}\mu\mathrm{h}^2)_{,a}h^a - (\rho + p)\dot{u}^a h_a = 0 \qquad (3.10)$$

and

$$(\rho - 3p)_{,a}h^a = 0 \qquad (3.11)$$

Now, for the divergence-free magnetic field (i.e., $h^a_{;a} = 0$), equations (2.7), (2.14), (3.10), and (3.11) yield

$$\rho_{,a}h^a = p_{,a}h^a = \mathrm{h}_{,a}h^a = 0 \tag{3.12}$$

Hence the proof of the theorem is complete.

4. *Killing Tensor of Second Order and the Perfect Magnetofluid*

We investigate the Killing tensor of second order in light of the perfect magnetofluid.

Theorem 4.1. If the three-space operator p_{ab} in the space-time permeated by the perfect magnetofluid is a Killing tensor then the energy density remains invariant along the flow vector.

Proof. The conditions (1.6) for p_{ab} to be a Killing tensor produce

$$3p_{(ab;c)} = -u_a u_{(b;c)} - u_b u_{(a;c)} - u_c u_{(a;b)} = 0 \tag{4.1}$$

since g_{ab} is redundant. From equation (4.1) we get

$$\theta = 0 \tag{4.2}$$

On using the equation of continuity (2.10) and equation (4.2) we have

$$\dot{\rho} = 0$$

Thus the proof of the theorem is complete.

Corollary 4.1. Equation (4.1) yields

$$\theta = 0, \qquad \dot{u}_a = 0, \qquad \sigma_{ab} = 0 \tag{4.3}$$

Thus, if the three-space projection operator p_{ab} is a Killing tensor, the flow is Killing.

Corollary 4.2. For the perfect magnetofluid with Killing three-space operator p_{ab}, the magnetic field is divergence-free and the isotropic pressure conserves along the magnetic field vector.

Proof. By condition $\dot{u}^a = 0$, equations (2.7) and (2.13) produce

$$h^a_{;a} = 0 \quad \text{and} \quad p_{,a}h^a = 0$$

These equations justify the statement.

Theorem 4.2. In the shear-free perfect magnetofluid two-space operator $q_{ab} = g_{ab} - u_a u_b + H_a H_b$ where $H^a = h^a/h$ is Killing if the stream lines are expansion-free.

Proof. With the conditions (1.6) for q_{ab} to be Killing, we have

$$H_a H_{(b;c)} + H_b H_{(c;a)} + H_c H_{(a;b)} - u_a u_{(b;c)} - u_b u_{(c;a)} - u_c u_{(a;b)} = 0 \tag{4.4}$$

Contraction of equation (4.4) yields

$$H_{a;b}H^b + H_a H^b_{;b} - \dot{u}_a - \theta u_a = 0 \tag{4.5}$$

The time component of equation (4.5) is

$$u_{a;b}H^a H^b + \theta = 0$$

i.e.,

$$\sigma_{ab}H^a H^b + \tfrac{2}{3}\theta = 0 \tag{4.6}$$

For shear-free perfect magnetofluid, equation (4.6) reduces to

$$\theta = 0$$

Hence the proof of the theorem is complete.

Theorem 4.3 If the symmetric stress-energy tensor of the perfect magnetofluid is a Killing tensor of the second order then the pressure, the energy density, and the magnitude of the magnetic field are constant.

Proof. In case of the symmetric stress-energy tensor (2.3), the condition $T_{(ab;c)} = 0$ yields

$$A_{,c}u_a u_b + A_{,b}u_c u_a + A_{,a}u_c u_a - B_{,c}g_{ab} - B_{,b}g_{ca} - B_{,a}g_{bc} + 2A[u_a u_{(b;c)} + u_b u_{(a;c)} + u_c u_{(a;b)}] - 2\mu[h_a h_{(b;c)} + h_b h_{(c;a)} + h_c h_{(a;b)}] = 0 \tag{4.7}$$

By various contractions of equation (4.7) we obtain

$$(\rho - 3p)_{,c} = 0 \tag{4.8}$$

$$(\rho + 2p + \tfrac{1}{2}\mu \mathrm{h}^2)_{,a}h^a = 0 \tag{4.9}$$

$$(\rho + \tfrac{1}{2}\mu \mathrm{h}^2)_{,a}u^a = 0 \tag{4.10}$$

$$Au_a + \tfrac{1}{2}(\rho + \tfrac{1}{2}\mu \mathrm{h}^2)_{,a} = 0 \tag{4.11}$$

$$B_{,a}h^a + \tfrac{1}{2}\mathrm{h}^2_{,a}h^a = 0 \tag{4.12}$$

From equations (4.8) to (4.12), we find that

$$\sigma_{,a} = p_{,a} = h_{,a} = 0 \tag{4.13}$$

Thus the proof of the theorem is complete.

Corollary 4.3. If the stress-energy tensor of the perfect magnetofluid is Killing, then stream-lines are expansion-free and the magnetic lines are divergence-free.

Proof. By virtue of (4.13) the equation of continuity (2.10) implies

$$\theta = 0, \qquad \text{as } \rho + p \neq 0 \tag{4.14}$$

while (2.13) yields

$$\dot{u}_a h^a = 0 \tag{4.15}$$

Consequently,

$$h^a_{;a} = 0$$

and the proof of the corollary is complete.

5. *Complexion Vector for the Perfect Magnetofluid*

The expression (1.9) for the perfect magnetofluid becomes

$$\alpha_{,a} = \frac{\eta^{cd}_{ab}(A^2 u^b_{;c} u_d + 2\mu A h^e u_{e;c} u^b h_d - \mu^2 \mathbf{h}^2 \mathbf{h}^b_{;c} h_d)}{\rho^2 + 3p^2 + \mu A h^2}$$

i.e.,

$$\alpha_{,a} = \frac{A^2 \omega_a + \eta^{cd}_{ab}(A\mu h^e u_{e;c} u^b h_d - \mu^2 h^2 h^b_{;c} h_d)}{\rho^2 + 3p^2 + \mu A \mathbf{h}^2} \tag{5.1}$$

Contracting with h^a, equation (5.1) produces

$$\alpha_{,a} h^a = \frac{A^2}{\rho^2 + 3p^2 + \mu A \mathbf{h}^2} \omega_a h^a \tag{5.2}$$

From equation (5.2) we get

$$\alpha_{,a} h^a = 0 \Leftrightarrow \omega_a h^a = 0 \qquad \text{as } A^2 \neq 0$$

Thus the complexion of the field in a perfect magnetofluid remains invariant along the magnetic lines if and only if these lines are normal to the vortex line.

Remark. The complexion of the space-time of the perfect magnetofluid with uniform magnetic field remains invariant along the world-line [cf (5.1)].

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